Contact symmetries of general linear second-order ordinary differential equations

1983 J. Phys. A: Math. Gen. 16 L455

(http://iopscience.iop.org/0305-4470/16/13/003)

J. Phys. A: Math. Gen. 16 (1983) L455–L460. Printed in Great Britain

LETTER TO THE EDITOR

Contact symmetries of general linear second-order ordinary differential equations

R Martini and P H M Kersten

Department of Applied Mathematics, Twente University of Technology, PO Box 217, 7500 AE Enschede, The Netherlands

Received 17 June 1983

Abstract. Using 1–1 mappings, the complete symmetry groups of contact transformations of general linear second-order ordinary differential equations are determined from two independent solutions of those equations, and applied to the harmonic oscillator with and without damping.

The complete Lie algebra of infinitesimal symmetries of a general linear second-order ordinary differential equation can be determined if we know two independent solutions explicitly. By a transformation described by Arnold (1983), there is a 1–1 mapping which transforms a linear differential equation

$$u_{tt}+a_1(t)u_t+a_2(t)u=0 \tag{1}$$

into

$$Y_{XX}=0, \tag{2}$$

i.e. by rectifying the solutions of (1).

The Lie point and Lie contact symmetries of (2) are (Lie 1893)

$$\partial_Y, X\partial_Y, Y\partial_Y, Y_X\partial_Y, XY_X\partial_Y, YY_X\partial_Y, X(Y-XY_X)\partial_Y, Y(Y-XY_X)\partial_Y \tag{3a}$$

and (Anderson and Ibragimov 1979)

$$(YG(Y-XY_X, Y_X)+H(Y-XY_X, Y_X))\partial_Y \tag{3b}$$

where in (3b) G and H are arbitrary functions of their variables. By using the inverse transformation, we transform the symmetries (3a), (3b) in a 1–1 manner into symmetries of (1) (Kumei and Bluman 1982).

Throughout this paper we shall use the local jet bundle formalism (Pirani *et al* 1979). All considerations are of local nature.

In two examples we consider the harmonic oscillator, clarifying the results of Wulfman and Wybourne (1976) and Schwarz (1983).

We start our discussion at the differential equation (1)

$$u_{tt}+a_1(t)u_t+a_2(t)u=0,$$

defined on $J^2(t,u)=\{(t,u,u_t,u_{tt})\}$. Let (2), i.e.

$$Y_{XX}=0,$$

be the differential equation defined on $J^2(X, Y)$. Let $u_1(t)$, $u_2(t)$ be two independent solutions of (1) such that

$$u_1(0)=0, \qquad u_2(0)\neq 0.$$

We now define the mapping $\phi: \{(t, u)\}\to\{(X, Y)\}$ by (Arnold 1983)

$$\bar{X}=u_1(t)/u_2(t), \qquad \bar{Y}=u/u_2(t); \tag{4}$$

since

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}t}=\frac{u_1'(t)u_2(t)-u_1(t)u_2'(t)}{u_2(t)^2}\bigg|_{t=0}\neq 0,$$

ϕ is an invertible mapping and

$$\phi^{-1}:\{(X, Y)\}\to\{(t, u)\}$$

is defined by

$$\bar{t}=[u_1/u_2]^{-1}(X), \qquad \bar{u}=Yu_2(\bar{t}). \tag{5}$$

The prolongation mapping of ϕ, leaving the contact ideal invariant (Pirani *et al* 1979),

$$p^1\phi:\{(t, u, u_t)\}\to\{(X, Y, Y_X)\},$$

is defined by

$$\bar{X}=\frac{u_1(t)}{u_2(t)}, \qquad \bar{Y}=\frac{u}{u_2(t)}, \qquad \bar{Y}_X=\frac{u_2(t)u_t-u_2'(t)u}{u_2(t)^2}\left[\frac{\mathrm{d}\bar{X}}{\mathrm{d}t}\right]^{-1}. \tag{6}$$

A somewhat tedious calculation shows that $p^2\phi: J^2(t, u)\to J^2(X, Y)$ is defined by (6) and

$$\bar{Y}_{XX}=u_2(t)(u_1'(t)u_2(t)-u_1(t)u_2'(t))^{-1}(u_{tt}+a_1(t)u_t+a_2(t)u), \tag{7}$$

from which it is clear (Kumei and Bluman 1982) that ϕ transforms solutions of (1) into solutions of (2) in a 1–1 manner.

We shall now derive the general formula for the transformation of vector fields. We shall only consider vertical vector fields, i.e.

$$V=F(X, Y, Y_X)\partial_Y+\dots \tag{8}$$

where the dots denote the prolongation terms, since every vector field

$$V^1=K(X, Y, Y_X)\partial_X+H(X, Y, Y_X)\partial_Y$$

is equivalent to (cf Kumei and Bluman 1982)

$$V^2=(H(X, Y, Y_X)-Y_XK(X, Y, Y_X))\partial_Y+\dots;$$

and since the prolonged field can be obtained from $F(X, Y, Y_X)$ by applying the total derivative operator D_X, we suppress the prolongation in the sequel. The mapping $(\phi^{-1})_*: T\{(X, Y)\}\to T\{(t, u)\}$ is defined by

$$((\phi^{-1})_*V)f(t, u)=V(f(\bar{t}, \bar{u}))|_{(t,u)}=F(\bar{X}, \bar{Y}, \bar{Y}_X)u_2(t)\,\partial f(t, u)/\partial u, \tag{9}$$

from which we obtain the general formula

$$(\phi^{-1})_*\{F(X, Y, Y_X)\partial_Y\}=F(\bar{X}, \bar{Y}, \bar{Y}_X)u_2(t)\partial_u. \tag{10}$$

So, by (10) infinitesimal symmetries of (2) are transformed by (10) into symmetries of (1) in a 1–1 manner.

Example 1. The harmonic oscillator is described by

$$u_{tt} + u = 0. \tag{11}$$

Two independent solutions of (11) are

$$u_1(t) = \sin(t), \qquad u_2(t) = \cos(t). \tag{12}$$

The mapping $p^1\phi : J^1(t, u) \to J^1(X, Y)$ is defined by

$$\bar{X} = \tan t, \qquad \bar{Y} = u/\cos t, \qquad \bar{Y}_X = u_t \cos t + u \sin t, \tag{13}$$

while $\phi^{-1} : \{(X, Y)\} \to \{(t, u)\}$ is given by

$$\bar{t} = \tan^{-1} X, \qquad \bar{u} = Y(1+X)^{-1/2}. \tag{14}$$

Application of (10) to the Lie point and Lie contact symmetries of (2), i.e. (3*a*) and (3*b*), yields table 1:

Table 1.

$Y_{XX} = 0$	$u_{tt} + u = 0$
∂_Y	$\cos t\, \partial_u$
$X\partial_Y$	$\sin t\, \partial_u$
$Y\partial_Y$	$u\partial_u$
$Y_X\partial_Y$	$(u_t \cos^2 t + \frac{1}{2}u \sin 2t)\partial_u$
$XY_X\partial_Y$	$(\frac{1}{2}u_t \sin 2t + u \sin^2 t)\partial_u$
$YY_X\partial_Y$	$(uu_t \cos t + u^2 \sin t)\partial_u$
$X(Y - XY_X)\partial_Y$	$(-u_t \sin^2 t + \frac{1}{2}u \sin 2t)\partial_u$
$Y(Y - XY_X)\partial_Y$	$(-uu_t \sin t + u^2 \cos t)\partial_u$
$(YG + H)\partial_Y$	$(uG(v, w) + \cos t H(v, w))\partial_u$

where in table1

$$v = u \cos t - u_t \sin t, \qquad w = u \sin t + u_t \cos t. \tag{15}$$

Using the relations

$$v^2 + w^2 = u^2 + u_t^2, \qquad t = \tan^{-1}(u/u_t) - \tan^{-1}(v/w),$$

it is a straightforward calculation to show that

$$uG(v, w) + \cos t H(v, w) = u\tilde{G}(v, w) + u_t\tilde{H}(v, w) \tag{16}$$

where $\tilde{G}(v, w)$, $\tilde{H}(v, w)$ are given by

$$\tilde{G}(v, w) = G(v, w) + [v/(v^2 + w^2)]H(v, w), \qquad \tilde{H}(v, w) = [w/(v^2 + w^2)]H(v, w),$$

where (16) is just the general form of Lie contact symmetries obtained by Schwarz (1983).

Example 2. The harmonic oscillator with damping is given by

$$u_{tt} + 2au_t + u = 0 \qquad (a > 0). \tag{17}$$

We have to distinguish three cases: (1) $a > 1$, (2) $a = 1$, (3) $0 < a < 1$.

(1) $(a>1)$.

Two independent solutions of (17) are

$$u_1(t)=e^{a_1t}-e^{a_2t}, \qquad u_2(t)=e^{a_2t}, \tag{18}$$

where

$$a_1=-a+(a^2-1)^{1/2}, \qquad a_2=-a-(a^2-1)^{1/2}.$$

Now, by (4) $p^1\phi: J^1(t,u)\to J^1(X,Y)$ is defined by

$$\bar{X}=e^{(a_1-a_2)t}-1, \qquad \bar{Y}=ue^{-a_2t}, \qquad \bar{Y}_X=(a_1-a_2)^{-1}(u_t-a_2u)e^{-a_1t}, \tag{19}$$

and $\phi^{-1}:\{(X,Y)\}\to\{(t,u)\}$ by

$$\bar{t}=(a_1-a_2)^{-1}\log(1+X), \qquad \bar{u}=Y(1+X)^{a_2/(a_1-a_2)}.$$

The complete Lie algebra of Lie point and Lie contact symmetries of (17) with $a>1$ is computed from (3a, b), (10), (19) and is given in table 2.1:

Table 2.1. $a>1$.

$Y_{XX}=0$	$u_{tt}+2au_t+u=0$
∂_Y	$\exp(a_2t)\partial_u$
$X\partial_Y$	$[\exp(a_1t)-\exp(a_2t)]\partial_u$
$Y\partial_Y$	$u\partial_u$
$Y_X\partial_Y$	$(a_1-a_2)^{-1}(u_t-a_2u)\exp[(a_2-a_1)t]\partial_u$
$XY_X\partial_Y$	$(a_1-a_2)^{-1}(u_t-a_2u)\{1-\exp[(a_2-a_1)t]\}\partial_u$
$YY_X\partial_Y$	$(a_1-a_2)^{-1}(uu_t-a_2u^2)\exp(-a_1t)\partial_u$
$X(Y-XY_X)\partial_Y$	$v[\exp(a_1t)-\exp(a_2t)]\partial_u$
$Y(Y-XY_X)\partial_Y$	$vu\partial_u$
$(YG+H)\partial_Y$	$[uG(v,w)+\exp(a_2t)H(v,w)]\partial_u$

where

$$v=(a_1-a_2)^{-1}[(a^1u-u_t)e^{-a_2t}-(a_2u-u_t)e^{-a_1t}], \qquad w=(a_1-a_2)^{-1}(u_t-a_2u)e^{-a_1t}.$$

(2) $(a=1)$.

The mapping $p^1\phi: J^1(t,u)\to J^1(X,Y)$ is defined by

$$\bar{X}=t, \qquad \bar{Y}=ue^t, \qquad \bar{Y}_X=(u+u_t)e^t. \tag{20}$$

The Lie point and Lie contact symmetries of (17) with $a=1$ are given in table 2.2:

Table 2.2. $a=1$.

$Y_{XX}=0$	$u_{tt}+2u_t+u=0$
∂_Y	$e^{-t}\partial_u$
$X\partial_Y$	$te^{-t}\partial_u$
$Y\partial_Y$	$u\partial_u$
$Y_X\partial_Y$	$(u+u_t)\partial_u$
$XY_X\partial_Y$	$t(u+u_t)\partial_u$
$YY_X\partial_Y$	$ue^t(u+u_t)\partial_u$
$X(Y-XY_X)\partial_Y$	$t(u-tu-tu_t)\partial_u$
$Y(Y-XY_X)\partial_Y$	$ue^t(u-tu-tu_t)\partial_u$
$(YG+H)\partial_Y$	$[uG(v,w)+e^{-t}H(v,w)]\partial_u$

where

$$v = (u - tu - tu_t)2^t, \qquad w = (u + u_t)e^t. \tag{21}$$

(3) $(0 < a < 1)$.

Two independent solutions of (17) are

$$u_1(t) = e^{-at} \sin bt, \qquad u_2(t) = e^{-at} \cos bt.$$

The mapping $p^1\phi : J^1(t, u) \to J^1(X, Y)$ in this case is defined by

$$\bar{X} = \tan bt \ \bar{Y} = \frac{u\,e^{at}}{\cos bt}, \qquad \bar{Y}_X = b^{-1}\,e^{at}(\cos bt)u_t + b^{-1}a\,e^{at}(\cos bt)u + e^{at}(\sin bt)u, \tag{22}$$

where $b = (1-a^2)^{1/2}$, and $\phi^{-1} : \{(X, Y)\} \to \{(t, u)\}$ by

$$\bar{t} = b^{-1} \tan^{-1} X, \qquad \bar{u} = e^{-a\bar{t}}(\cos b\bar{t})Y.$$

The Lie point and Lie contact symmetries of (17) with $0 < a < 1$ have been computed by (3), (10), (22) and are given in table 2.3:

Table 2.3. $0 < a < 1$, $b = (1-a^2)^{1/2}$.

$Y_{XX} = 0$	$u_{tt} + 2au_t + u = 0$
∂_Y	$e^{-at}\cos bt\partial_u$
$X\partial_Y$	$e^{-at}\sin bt\partial_u$
$Y\partial_Y$	$u\partial_u$
$Y_X\partial_Y$	$b^{-1}(u_t \cos^2 bt + au \cos^2 bt + \frac{1}{2}bu \sin 2bt)\partial_u$
$XY_X\partial_Y$	$b^{-1}(\frac{1}{2}u_t \sin 2bt + \frac{1}{2}au \sin 2bt + bu \sin^2 bt)\partial_u$
$YY_X\partial_Y$	$b^{-1}\,e^{at}(uu_t \cos bt + au^2 \cos bt + bu^2 \sin bt)\partial_u$
$X(Y - XY_X)\partial_Y$	$b^{-1}(-u_t \sin^2 bt - au \sin^2 bt + \frac{1}{2}bu \sin 2bt)\partial_u$
$Y(Y - XY_X)\partial_Y$	$b^{-1}\,e^{at}(-uu_t \sin bt - au^2 \sin bt + bu^2 \cos bt)\partial_u$
$(YG + H)\partial_Y$	$[uG(v, w) + e^{-at} \cos bt H(v, w)]\partial_u$

where

$$v = b^{-1}\,e^{at}(-u_t \sin bt - au \sin bt + bu \cos bt),$$

$$w = b^{-1}\,e^{at}(u_t \cos bt + au \cos bt + bu \sin bt).$$

For $a = 0$, $b = 1$; table 2.3 reduces to table 1.

In conclusion, using 1–1 mappings, which transform differential equations into $Y_{XX} = 0$, we are able to construct the complete Lie algebra of Lie contact symmetries for linear second-order differential equations.

References

Anderson R L and Ibragimov N H 1979 *Lie Bäcklund Transformations in Applications, Siam Stud. Appl. Math.* (Philadelphia)

Arnold V I 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* ch 1, § 6 (Grundlehren der Mathematischen Wissenschaften 250) (New York: Springer)

Kumei S and Bluman G W 1982 *SIAM J. Appl. Math.* **42** 1157–73

Lie S 1893 *Vorlesungen über continuierliche Gruppen* (Leipzig: Teubner) (1971 New York: Chelsea)
Pirani F A E, Robinson D C and Shadwick W F 1979 *Local Jet Bundle Formulation of Bäcklund Transformations, Mathematical Physics Studies* 1 (Dordrecht: Reidel)
Schwarz F 1983 *J. Phys. A: Math. Gen.* **16** L133–5
Wulfman C E and Wybourne B G 1976 *J. Phys. A: Math. Gen.* **9** 507–18